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Orthogonality Between Scales in a Renormalization Group for Fermions

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Having in mind the development of a technical tool to treat fermionic systems, we propose a Kadanoff–Wilson block renormalization transformation employing unusual averages (an inevitable artifact due to the specificity of lattice fermions and to the desired transformation properties). The free propagator is decomposed into operators associated to different momentum scales and with orthogonal relations, and the effective actions generated from the Dirac operator by the transformations present uniform exponential decay. We argue to show the usefulness of the formalism to study correlation functions of interacting fermions.

KEY WORDS: Orthogonality between scales; renormalization group; fermions.

1. INTRODUCTION

Renormalization group (RG) techniques have been used as a successful tool for rigorous analysis of several fields: problems ranging from classical mechanics to quantum many-body systems have been treated via such an approach. A large and useful formalism has been developed,[1,2] although usually quite intricate (leading to the search for simplifications).

Recently,[3,4] studying the well-known lattice dipole gas [and $(\nabla\phi)^4$ models] with the RG techniques already developed in ref. 1 but emphasizing the property of orthogonality between different momentum scales in the transformation (property associated to the wavelets implicit in the structure of the block RG), we established exact and simple formulas for the correlation functions with a good control of the dominant and subdominant terms. The nonproliferation of terms and the simplicity of the

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final formulas obtained in that work showed us the usefulness of the "orthogonality of scales" property to technically improve the RG formalism.

Concerning the fermionic models, however, the implementation of a similar RG transformation (i.e., a Kadanoff–Wilson block-spin RG with the orthogonal property) presents some problems. We recall that even the formulation of lattice fermionic theories is troublesome: the doubling of the free spectrum due to a naive discretization of the Dirac equation is well known.

The flow of the Wilson action via a block RG transformation has been rigorously studied in ref. 5: using a transformation with a Gaussian weight function (which breaks chiral symmetry explicitly), the fixed point is obtained and the locality of the effective actions (uniform exponential decay for the actions rescaled to the unitary lattice) is shown as well as other useful results (such as the telescopic decomposition of the free propagator—details in the next section). However, the orthogonality between scales is lost, the property responsible for the simplicity of the correlation formulas for interacting systems, as said above. Also, in ref. 5, using a transformation with a δ weight function (which "abruptly" separates the scales leading to the orthogonal property), the unexpected fact is shown that the effective actions do not maintain the uniform exponential decay, which, unfortunately, makes the transformation inadequate to treat interacting fermions.

Thus, having in mind the development of RG techniques and based on the search for "technical simplicity," we propose in this paper an RG transformation for lattice fermionic models that (initially) applied to the free action gives us a telescopic decomposition of the free propagator in terms of operators with the property of orthogonality between scales, and that also makes local all the effective actions (kernels with uniform exponential decay). We hope later, using this orthogonal property and the local actions, to study interacting fermions. Specifically, we believe that this RG transformation will make easier the study of correlation functions (see Section 5).

The rest of the paper is organized as follows. In Section 2 we introduce some definitions, review some recent results, and state the theorem about the uniform exponential decay of the effective actions (and other operators). In Section 3 we present an RG transformation with δ weight function but with the average over blocks of spins given by a complex (imaginary) perturbation of the usual one (and also prove that any real perturbation does not lead to a transformation with the required properties). Section 4 is devoted to technical proofs of decay properties, and Section 5 to final comments.

2. DEFINITIONS AND RECENT RESULTS

In refs. 3 and 4, via block RG techniques and (emphatically) using the "orthogonality of scales" property, we study the correlation functions of lattice scalar field models such as $(\nabla\phi)^4$ and dipole gas, obtaining exact and simple formulas which separate the dominant and subdominant terms and make clear the long-distance behavior. Considering models described by interactions on unitary finite lattices (with L^{Nd} points; the thermodynamic limit is considered later) such as

$$\mathscr{H}(\phi) = \tfrac{1}{2} b_0(\phi, \Delta\phi) + V(\phi) \tag{2.1}$$

where $\phi(x) \in \mathbf{R}$, $x \in \Lambda_N \subset \mathbf{Z}^d$, $d \geqslant 3$, $\Delta \equiv \partial^\dagger \partial$ (for Dirichlet boundary conditions, otherwise plus a reguralizer), V a function of $\partial_\mu \phi(x)$, and using the block RG transformation

$$\exp[-\mathscr{H}^1(\psi)] = \int \exp[-\mathscr{H}(\phi)]\, \delta(C\phi - \psi)\, D\phi \tag{2.2}$$

where $\psi \in \mathbf{R}^{\Lambda_{N-1}}$, $D\phi = \prod_{x \in \Lambda_N} d\phi(x)$, $\delta(C\phi - \psi) = \prod_{x \in \Lambda_{N-1}} \delta(C\phi(x) - \psi(x))$, with $C\phi(x)$ meaning the rescaled average (canonical scaling) over blocks b^L_{Lx} of size L, centered in $Lx \in \Lambda_N$,

$$C\phi(x) = L^{(d-2)/2} L^{-d} \sum_{y \in b^L_{Lx}} \phi(y) \tag{2.3}$$

we follow the flow of the generating function $Z(h) \equiv \int \exp[-\mathscr{H}(\phi) + (h, \phi)]\, D\phi$, obtaining, after n steps of the RG transformation ($n \leqslant N$),

$$Z(h) = c \exp[\tfrac{1}{2}(h, P_n h)] \times \int \exp\{-V^n(\partial_\mu[M_n \phi + G_n h]) - \tfrac{1}{2} b_n(\phi, \Delta_n \phi)\}\, D\phi \tag{2.4}$$

where c does not depend on h; b_n is the wavefunction renormalization constant at step n; V^n is the nth irrelevant perturbative potential (the potential without its marginal quadratic part); the propagators P_n and G_n are written in terms of operators describing interactions in different momentum scales (and associated with a telescopic decomposition of the free propagator and the lattice wavelets[3,4]); Δ_n is a "local" effective action (exponential decay), which goes with n to the Gaussian fixed point; and M_n is the n-step minimizer (also with exponential decay).

Once more, we emphasize the simplicity of the formula obtained for the generating function: due to the orthogonal property there is no mix between different momentum scales (see the expressions for P_n and G_n in refs. 3 and 4).

Turning to the fermionic systems, we observe that the implementation of a similar RG transformation with the properties related above is not, say, immediate.

Free lattice (and continuous) Euclidean fermions are treated in ref. 5 using block RG transformations with a Gaussian and also a δ weight function, obtaining expressions analogues to those described above. The considered actions, living on lattices with spacing ε (initially), are given by the ε-lattice Wilson version of the Dirac operator,

$$D = \sum_{\mu=1}^{d} \gamma_\mu \left(\frac{\partial_\mu^\varepsilon - \partial_\mu^{\varepsilon\dagger}}{2} \right) - \frac{1}{2} \varepsilon \Delta^\varepsilon, \qquad \Delta^\varepsilon = \sum_{\mu=1}^{d} \frac{1}{\varepsilon} (\partial_\mu^\varepsilon + \partial_\mu^{\varepsilon\dagger}) \tag{2.5}$$

where ∂_μ^ε is the ε-lattice forward derivative, $\partial_\mu^{\varepsilon\dagger}$ is its adjoint, and γ_μ are anti-Hermitian Dirac matrices obeying $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$. The extra term (breaking chiral symmetry) is introduced to suppress the doubler fermions and vanishes in the continuous limit ($\varepsilon \to 0$). The flow of the free action is studied via the RG transformation $T_{a,L}^\varepsilon$ defined as

$$\begin{aligned} \exp(\bar{\chi}, D_1 \chi) &\equiv [T_{a,L}^\varepsilon \exp(\cdot, D\cdot)](\bar{\chi}, \chi) \\ &= N \int d\bar{\psi}\, d\psi \exp[a(L\varepsilon)^{-1} (\bar{\chi} - Q\bar{\psi}, \chi - Q\psi)] \exp(\bar{\psi}, D\psi) \end{aligned} \tag{2.6}$$

where $\bar{\psi}, \psi$ ($\bar{\chi}, \chi$) are independent Grassmann algebra generators (with suppressed spinor and lattice indices), and mean ε ($L\varepsilon$) lattice fields. Q is the usual arithmetic averaging operator over a block of side size $L\varepsilon$, and a is a real, positive parameter (in the limit $a \to \infty$ we have the RG transformation with δ weight function). N is a normalization constant such that

$$\int \exp[\bar{\chi}, D_1 \chi]\, d\bar{\chi}\, d\chi = \int \exp[\bar{\psi}, D\psi]\, d\bar{\psi}\, d\psi$$

Successive RG transformations are introduced obeying the semigroup property

$$T_{a,L}^{L^{k-1}\varepsilon} T_{a,L}^{L^{k-2}\varepsilon} \cdots T_{a,L}^{\varepsilon} = T_{a_k, L^k}^{\varepsilon}$$

where T_{a_k, L^k}^ε is defined as in (2.6) with $a_k = [(1 - L^{-1})/(1 - L^{-k})]a$ replacing a, L^k replacing L, and Q_k (arithmetic averaging operators over blocks of side $L^k\varepsilon$) replacing Q. Irrespective of the domain lattice (i.e., of the lattice spacing), we use the same symbol for the arithmetic averages. With these transformations, after simple algebraic manipulations, the telescopic decomposition of the free propagator is obtained:

$$\begin{aligned} D^{-1} &= \sum_{j=0}^{n-1} [D^{-1}Q_j^\dagger D_j Q_j D^{-1} - D^{-1}Q_{j+1}^\dagger D_{j+1} Q_{j+1} D^{-1}] \\ &\quad + D^{-1}Q_n^\dagger D_n Q_n D^{-1} \\ &= \sum_{j=0}^{n-1} M_j \Gamma_j M_j^\dagger + M_n D_n^{-1} M_n^\dagger \end{aligned} \tag{2.7}$$

where $D_0 \equiv D$, $Q_0 \equiv I$; and $M_j = D^{-1}Q_j^\dagger D_j$, $M_j^\dagger = D_j Q_j D^{-1}$ are $L^j\varepsilon$ (ε) to ε ($L^j\varepsilon$) lattice operators; $\Gamma_j = D_j^{-1} - D_j^{-1} Q^\dagger D_{j+1} Q D_j^{-1}$ is an $L\varepsilon$ lattice operator. It is shown that the kernels of M_j, $M_j^\dagger$, and Γ_j have exponential decay such that

$$M_j \Gamma_j M_j^\dagger (x, x') \sim \frac{1}{(L^j\varepsilon)^{d-1}} \exp\left(-\frac{1}{L^j\varepsilon}|x-x'|\right), \qquad x, x' \in \varepsilon\mathbf{Z}^d \tag{2.8}$$

that is, (2.7) gives us a decomposition into momentum scales $(L^j\varepsilon)^{-1}$.

Rescaling the operators (after k steps) to the unitary lattice, it is shown (Theorem III.1 in ref. 5) that, for a small, the following result holds:

Theorem 2.1. $\exists\beta > 0$, $c > 0$ independent of k such that

$$\begin{aligned} |D_{(k)}(x, x')| &\leqslant c \exp[-\beta\,|x-x'|] \\ |\Gamma_{(k)}(x, x')| &\leqslant c \exp[-\beta\,|x-x'|] \\ |D_{(\eta)}^{-1} Q_k^\dagger D_k(y, x)| &\leqslant c \exp[-\beta\,|y-x|] \\ |D_{(k)} Q_k D_{(k)}^{-1}(x, y)| &\leqslant c \exp[-\beta\,|x-y|] \end{aligned}$$

for $y, y' \in L^{-k}\mathbf{Z}^d$, $x, x' \in \mathbf{Z}^d$; $\Gamma_{(k)}$, $D_{(k)}$ in the unitary lattice, and $D_{(\eta)}$ the Dirac operator (Wilson version) in the lattice $\eta = L^{-k}$.

This theorem is proved in Section 4 for the new RG to be proposed. The problem with the RG transformation with finite a is that we have

$$D_{(k)} = a_k(I + a_k Q_k D_{(\eta)}^{-1} Q_k^\dagger)^{-1} \tag{2.9}$$

[formula (3.11) in ref. 5] and there is no orthogonality between the operators $\{M_j \Gamma_j M_j^\dagger\}|_{j=1}^{n-1}$, $M_n D_n^{-1} M_n^\dagger$ in the "norm" $(\cdot, D\cdot)$ [which may be checked using the formulas below (2.7)]. For infinite a, i.e., considering the RG transformation with the δ weight function, the effective actions become

$$D_{(k)} = (Q_k D_{(\eta)}^{-1} Q_k^\dagger)^{-1} \tag{2.10}$$

following the orthogonal property, but losing the uniform exponential decay of these actions (i.e., invalidating Theorem 2.1), as proved in Section 4 of ref. 5 (more details below), which should lead to a "local" fixed point.

Thus, to treat fermions, we are forced to develop a more elaborate formalism.

3. UNUSUAL AVERAGES FOR BLOCK RG TRANSFORMATIONS

The property of orthogonality between scales, as we have noted, seems to be associated with the RG transformation with δ weight function, which is intuitively expected since the δ function recalls separation in a sharp manner. Another way to realize an orthogonal decomposition into different momentum scales is to introduce characteristic functions in the Fourier transform of D^{-1}, but this leads to operators without exponential decay in position space (and so does not interest us).

The algebraic structures of the operators related to the "δ-function" transformation may be obtained by direct inspection, and are given by formulas (2.7) and (2.10). Note that the operators depend only on the initial action D and on the average Q (besides, of course, on the RG weight function). Thus, we shall investigate unusual averages in a transformation with δ weight function.

We define general averages Q over blocks of "side L" (although, for one block, sities outside it may also contribute to the average) as

$$\begin{aligned} Qf(u) &\equiv \sum_x W(x)\, f(Lu - x), \qquad x, u \in \xi \mathbf{Z}^d \\ &= \frac{1}{(2\pi)^d} \int \tilde{W}(p)\, \tilde{f}(p) e^{ip \cdot Lu}\, dp \end{aligned} \tag{3.1}$$

where $p \in (-\pi/\xi, \pi/\xi]^d$, and $\tilde{W}(p)$ is the Fourier transform of $W(x)$, to be properly chosen [the same for $\tilde{f}(p)$]. Writing $p = p'/L + l/L$, where $|p'_\mu| \leqslant \pi/\xi$, $l = 2\pi m/\xi$, $|m_\mu| < L/2$, i.e., $m \in \mathbf{Z}^d \cap (-L/2, L/2)^d$, we have

$$(Qf)^\sim (p') = \sum_l \frac{1}{L^d} \tilde{W}\left(\frac{p'}{L} + \frac{l}{L}\right) \tilde{f}\left(\frac{p'}{L} + \frac{l}{L}\right) \tag{3.2}$$

and for the adjoint,

$$(Q^{\dagger}f)^{\sim}\left(\frac{p'}{L}+\frac{l}{L}\right)=\bar{\tilde{W}}\left(\frac{p'}{L}+\frac{l}{L}\right)\tilde{f}(p') \tag{3.3}$$

where $\bar{\tilde{W}}$ means the complex conjugate to $\tilde{W}$.

Hence, for the inverse of effective action,

$$\begin{aligned}(QD^{-1}Q^{\dagger}f)^{\sim}\,(p')&=\sum_{l}\frac{1}{L^{d}}\left|\tilde{W}\left(\frac{p'}{L}+\frac{l}{L}\right)\right|^{2}\tilde{D}^{-1}\left(\frac{p'}{L}+\frac{l}{L}\right)\tilde{f}(p')\\&=\tilde{D}_{1}^{-1}(p')\,\tilde{f}(p')\end{aligned} \tag{3.4}$$

To obtain the expression for $(Q_nD^{-1}Q_n^{\dagger})^{-1}$ we check the relation between Q_n and Q. Due to the semigroup property of an RG transformation it follows that $Q_{k+j}=Q_kQ_j$. This property (and, of course, the formulas due to the RG with δ weight function) leads to the orthogonality between scales: taking $M_j\Gamma_jM_j^{\dagger}$ and $M_i\Gamma_iM_i^{\dagger}$, $j\neq i$, as an example, from the formulas below (2.7) we have $M_j\Gamma_jM_j^{\dagger}DM_i\Gamma_iM_i^{\dagger}=0$. Thus,

$$\begin{aligned}Q_nf(u)&=\sum_{x}W_n(x)\,f(L^nu-x)\\&=\sum_{x_1,\dots,x_n}W(x_1)\cdots W(x_n)\,f(L^nu-L^{n-1}x_1-L^{n-2}x_2-\cdots x_n)\end{aligned} \tag{3.5}$$

(Q_n meaning average over blocks "size L^n"). Hence,

$$W_n(L^{n-1}x_1+L^{n-2}x_2+\cdots+x_n)=W(x_1)\cdots W(x_n) \tag{3.6}$$

and

$$\tilde{W}_n(p)=\sum_{x_1}W(x_1)e^{-ip\cdot L^{n-1}x_1}\times\cdots\sum_{x_n}W(x_n)e^{-ip\cdot x_n}$$

Writing for the term with x_1

$$p=\frac{p_1+l_1}{L^{n-1}},\qquad p_1\in\left(-\frac{\pi}{\xi},\frac{\pi}{\xi}\right]^d,\qquad l_1=\frac{2\pi m_1}{\xi},\qquad |m_1|<\frac{L^{n-1}}{2}$$

we have

$$\sum_{x_1}W(x_1)e^{-ip\cdot L^{n-1}x_1}=\sum_{x_1}W(x_1)e^{-ip_1\cdot x_1}=\tilde{W}(p_1)$$

(where $p_1=L^{n-1}p-l_1$). With similar considerations,

$$\tilde{W}_n(p)=\tilde{W}(p_1)\,\tilde{W}(p_2)\cdots\tilde{W}(p_{n-1})\,\tilde{W}(p)=\tilde{W}(L^{n-1}p)\,\tilde{W}(L^{n-2}p)\cdots\tilde{W}(p) \tag{3.7}$$

[since $\tilde{W}(p)$ has period $2\pi/\xi$]. From (3.4), in terms of $\tilde{W}_n$ the inverse of the effective action after n steps becomes (rescaled to the unitary lattice)

$$\tilde{D}_{(n)}^{-1}(p) = \sum_{l=2\pi m} \frac{1}{L^{nd}} \left| \tilde{W}_n\left(\frac{p+l}{L^n}\right)\right|^2 \tilde{D}^{-1}\left(\frac{p+l}{L^n}\right), \qquad |m_\mu| < \frac{L^n}{2} \tag{3.8}$$

For the usual rescaled average [i.e., $W(x) = L^\alpha$ for $x \in b_0^L$, α related to the field dimension], using (3.7), we have

$$|\tilde{W}_n(p)|^2 = L^{2n\alpha} \prod_{\mu=1}^{d} \frac{\sin^2(p_\mu L^n/2)}{\sin^2(p_\mu/2)} \tag{3.9}$$

and

$$\tilde{D}_\infty^{-1}(p) = \sum_{l \in 2\pi \mathbf{Z}^d} \prod_{\mu=1}^{d} \frac{\sin^2(p_\mu/2)}{[(p_\mu + l_\mu)/2]^2} \times \frac{-i\gamma \cdot (p+l)}{(p+l)^2} \tag{3.10}$$

for $\alpha = -(d+1)/2$.

Now, from (3.10), we note the problem with the effective actions as indicated before. $\tilde{D}_\infty^{-1}$ is a periodic function (Fourier transform of a function in the unitary lattice) with period 2π, so $\tilde{D}_\infty^{-1}|_\mu (p_\mu = -\pi) = \tilde{D}_\infty^{-1}|_\mu (p_\mu = \pi)$. But it is easy to see that $\tilde{D}_\infty^{-1}$ is also an odd function $[\tilde{D}_\infty^{-1}|_\mu (p_\mu = -\pi) = -\tilde{D}_\infty^{-1}|_\mu (p_\mu = \pi)]$. Thus $\tilde{D}_\infty^{-1}|_\mu (p_\mu = \pi) = 0$, and the inverse $(\tilde{D}_\infty)$ does not even exist at these points (anyway, we abusively maintain the notation $\tilde{D}_\infty^{-1}$). Before trying to solve this problem considering other averages, we prove a simple but important result.

Lemma 3.1. For any real $W(x)$, $x \in \mathbf{Z}^d$, the corresponding $\tilde{D}_\infty^{-1}|_\mu$ vanishes at $p_\mu = \pi$.

Proof. For real $W(x)$,

$$\begin{aligned}\tilde{W}(p) &= \sum_{x \in \mathbf{Z}^d} W(x) \cos(p \cdot x) - i \sum_{x \in \mathbf{Z}^d} W(x) \sin(p \cdot x) \\ &\equiv A(p) - iB(p)\end{aligned}$$

with $A(p)$ and $B(p)$ real, A an even function, and B odd. Hence, $|\tilde{W}(p)|^2 = A^2(p) + B^2(p)$ is an even function, which is enough to prove the lemma: from (3.8) and (3.7),

$$\tilde{D}_\infty^{-1}(p) = \sum_{l \in \mathbf{Z}^d} |f(p+l)|^2 \times \frac{-i\gamma \cdot (p+l)}{(p+l)^2}$$

with f (function of $\tilde{W}$) even, and so $\tilde{D}_\infty^{-1}$ odd with period 2π, i.e., $\tilde{D}_\infty^{-1}|_\mu (p_\mu = \pi) = 0$. ∎

This lemma says that the consideration of complex averages is a must.

Thus, we propose to investigate the average given by the usual one [i.e., $W(x)$ constant for x inside the block b_0^L with L^d points and centered at zero, and vanishing outside it] plus a small complex perturbation properly chosen to break the undesirable symmetry of the fixed point (which makes $\tilde{D}_{\infty}^{-1}$ vanish, as described above). We take, for the Fourier transform of $W(x)$, x in the unitary lattice, L odd,

$$\tilde{W}(p) = L^{\alpha} \prod_{\mu=1}^{d} \left\{ (1+\delta \sin p_\mu) \times \sum_{|x_\mu| < L/2} 1 \times e^{-ip_\mu x_\mu} \right\} \tag{3.11}$$

($\delta = 0$ gives the usual formula). In the position space, $W(x) \equiv L^{\alpha} \prod_{\mu=1}^{d} W_\mu(x_\mu)$ with

$$\begin{aligned} W_\mu(x_\mu) &= i\delta/2, && x_\mu = -(L-1)/2 - 1 \\ &= 1 + i\delta/2, && x_\mu = -(L-1)/2 \\ &= 1, && x_\mu = -(L-1)/2 + m, \quad m \in \{1, 2,..., L-2\} \\ &= 1 - i\delta/2, && x_\mu = (L-1)/2 \\ &= -i\delta/2, && x_\mu = (L-1)/2 + 1 \end{aligned} \tag{3.12}$$

Note that the perturbation changes, in relation to the usual average, only points in the boundary of a block (and that the average of one block considers even sities outside it).

Now, starting with the Wilson action, we get

$$\begin{aligned} \tilde{D}_{(n)}^{-1}(p) = \sum_{\substack{|m_\mu| < L^n/2 \\ l = 2\pi m \in 2\pi \mathbf{Z}^d}} L^{(2\alpha+1-d)n} \prod_{\mu=1}^{d} \left\{ \left| 1 + \delta \sum_{k=1}^{n} \sin\left(\frac{[p_\mu + l_\mu]}{L^k} \right) + \mathcal{O}(\delta^2) \right|^2 \right. \\ \left. \times \frac{\sin^2([p_\mu + l_\mu]/2)}{\sin^2([p_\mu + l_\mu]/2L^n)} \right\} \frac{-i\gamma \cdot (p+l)}{(p+l)^2} \end{aligned} \tag{3.13}$$

that is, an effective action still periodic (with period 2π), but no longer an odd function (unitarity is also lost), which avoids the vanishing at $p_\mu = \pi$. Observe also that the new action has not been changed at $p = 0$.

The next section is devoted to proving that this (technical) small perturbation is enough to give us an RG transformation with the required properties: orthogonality between scales and locality of the effective actions (besides locality of the fluctuation field two-point function Γ_j and the minimizer M_j, etc.).

4. DECAY PROPERTIES

In this section we prove the theorem of Section 2 for the RG transformation proposed here, that is, we establish the uniform exponential decay for the effective actions $D_{(j)}$, fluctuation two-point functions $\Gamma_{(j)}$, and minimizers $M_{(j)}$. The proofs are carried out showing boundedness and analyticity of the Fourier transform of the operators in a small complex strip

$$T_c = \{|\mathrm{Im}(p_1)| < \alpha',\ \mathrm{Re}(p_1,..., p_d) \in (-\pi, \pi]^d\}$$

resulting in exponential decay in the $\mu = 1$ direction and, by symmetry, in all directions.

Considering the initial lattice in $\varepsilon\mathbf{Z}^d$, we write the average $\tilde{W}_j(p)$ as (for the final lattice in $L^j\varepsilon\mathbf{Z}^d$)

$$\tilde{W}_j(p) = \prod_{\mu=1}^{d} \mathscr{C}^{\varepsilon}_{j,\mu}(p)\, \tilde{w}_j(\varepsilon p_\mu) \tag{4.1}$$

where $\tilde{w}_j(\cdot)$ is the usual average (without the scaling factor) and $\mathscr{C}^{\varepsilon}_{j,\mu}(p) = \mathscr{C}_j(\varepsilon p_\mu)$ the perturbation $[\mathscr{C}_1 \equiv \mathscr{C},\ \mathscr{C}_\mu(p) = 1 + \delta \sin p_\mu$; see (3.11), and (3.7) for the relation between $\mathscr{C}_j$ and $\mathscr{C}]$. We have, for $g \in l_2(L^j\varepsilon\mathbf{Z}^d)$, $f \in l_2(\varepsilon\mathbf{Z}^d)$, and $\partial^{\alpha}_{\mu}(p) \equiv (e^{i\alpha p_\mu} - 1)/\alpha$,

$$(Q_j f)^{\sim}(p) = \sum_{l} \prod_{\mu=1}^{d} \mathscr{C}^{\varepsilon}_{j,\mu}(p+l) \frac{\partial^{L^j\varepsilon}_{\mu}(p+l)}{\partial^{\varepsilon}_{\mu}(p+l)} \tilde{f}(p+l) \tag{4.2}$$

$$(Q^{\dagger}_j g)^{\sim}(p+l) = \prod_{\mu=1}^{d} \overline{\mathscr{C}^{\varepsilon}_{j,\mu}}(p+l) \overline{\left(\frac{\partial^{L^j\varepsilon}_{\mu}(p+l)}{\partial^{\varepsilon}_{\mu}(p+l)}\right)}\, \tilde{g}(p) \tag{4.3}$$

where

$$p \in \left(-\frac{\pi}{L^j\varepsilon}, \frac{\pi}{L^j\varepsilon}\right]^d, \qquad l = \frac{2\pi m}{L^j\varepsilon}, \qquad m \in \mathbf{Z}^d$$

such that

$$p + l \in \left(-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d, \qquad L \text{ odd}$$

For the Dirac operator (Wilson version),

$$D_{(\eta)}(p) = i \sum_{\mu=1}^{d} \gamma_\mu \frac{\sin \eta p_\mu}{\eta} + \eta \sum_{\mu=1}^{d} \frac{1 - \cos \eta p_\mu}{\eta^2} \equiv i\gamma \cdot K(p) + M(p) \tag{4.4}$$

$$D^{-1}_{(\eta)}(p) = \frac{-i\gamma \cdot K(p) + M(p)}{K^2(p) + M^2(p)} \tag{4.5}$$

and for the other operators we get the following result:

Lemma 4.1. (a)

$$D_{(k)}(x, x') = \int \frac{dp}{(2\pi)^d} D_{(k)}(p) e^{ip\cdot(x-x')}$$
$$= \int \frac{dp}{(2\pi)^d} \left(\sum_l D_{(\eta)}^{-1}(p+l) \prod_{\mu=1}^{d} \left\{ \left| \mathscr{C}_{k,\mu}\left(\frac{p+l}{L^k}\right) \right.\right.\right.$$
$$\left.\left.\left. \times \frac{\partial_\mu^1(p+l)}{\partial_\mu^\eta(p+l)} \right|^2 \right\} \right)^{-1} e^{ip\cdot(x-x')}$$

(b)

$$\Gamma_{(k)}(x, x') = \sum_{l\neq l'} \int_{(-\pi/L,\, \pi/L]^d} \frac{dp'}{(2\pi)^d} D_{(k)}^{-1}(p'+l')\, D_{(k+1),\, L}(p')\, D_{(k)}^{-1}(p'+l)$$
$$\times \left\{ \prod_\mu \left| \mathscr{C}_\mu(p'+l) \frac{\partial_\mu^L(p'+l)}{\partial_\mu^1(p'+l)} \right|^2 \right.$$
$$- \prod_\mu \left[\overline{\mathscr{C}_\mu}(p'+l') \overline{\left(\frac{\partial_\mu^L(p'+l')}{\partial_\mu^1(p'+l')} \right)} \mathscr{C}_\mu(p'+l) \right.$$
$$\left. \left. \times \frac{\partial_\mu^L(p'+l)}{\partial_\mu^1(p'+l)} \right] e^{-i(l-l')\cdot x'} \right\} e^{il\cdot(x-x')} e^{ip'\cdot(x-x')}$$

(c)

$$D_{(\eta)}^{-1} Q_k^\dagger D_{(k)}(y, x) = \int \frac{dp}{(2\pi)^d} \sum_l \prod_{\mu=1}^{d} \left\{ \overline{\mathscr{C}_{k,\mu}}\left(\frac{p+l}{L^k}\right) \overline{\left(\frac{\partial_\mu^1(p+l)}{\partial_\mu^\eta(p+l)}\right)} \right\}$$
$$\times D_{(\eta)}^{-1}(p+l)\, D_{(k)}(p) e^{i(p+l)\cdot y} e^{-ip\cdot x}$$

for $x, x' \in \mathbf{Z}^d$, $y \in L^{-k}\mathbf{Z}^d$, $p \in (-\pi, \pi]^d$, $l = 2\pi m$, $m \in \mathbf{Z}^d$ such that $p + l \in (-L^k\pi, L^k\pi]^d$ in (a), (c); $p \in (-\pi/L, \pi/L]^d$, $l, l' = 2\pi m/L$, $m \in \mathbf{Z}^d$ such that $p + l$, $p + l' \in (-\pi, \pi]^d$ in (b).

Proof. Immediate for (a) and (c). For (b) we use the momentum-space representation of Γ_k [expression below (2.7)], write the first term D_k^{-1} as

$$D_k^{-1}(p+l) = D_k^{-1}(p+l)\, D_{k+1}(p)\, D_{k+1}^{-1}(p)$$
$$= D_k^{-1}(p+l)\, D_{k+1}(p)[Q D_k^{-1} Q^\dagger(p)] \qquad \text{[see (2.24)]}$$

and use the expressions (4.2) and (4.3) for Q and $Q^\dagger$ above. Then, some manipulations lead to the final formula. ∎

We obtain the necessary bounds for the Fourier transform of the operators in the theorem of Section 2 after separating a factor $D_{(\eta)}$ in the integrands of Lemma 4.1 (in order to treat possible singularities).

Theorem 4.1. (a)

$$D_{(k)}(p) = U^{-1}(p)\, D_{(\eta)}(p)$$

$$U(p) = D_{(\eta)}(p) \sum_l D_{(\eta)}^{-1}(p+l) \prod_{\mu=1}^{d} \left| \mathscr{C}_{k,\mu}\left(\frac{p+l}{L^k}\right) \frac{\partial_\mu^1(p+l)}{\partial_\mu^\eta(p+l)} \right|^2$$

$U(p)$ analytic in T_c, $p \in (-\pi, \pi]^d$, $l = 2\pi m$, $m \in \mathbf{Z}^d$, $p + l \in (-L^k\pi, L^k\pi]^d$,

$$|U(p)| < c, \qquad |U^{-1}(p)| < c', \qquad p \in T_c$$

(b)

$$D_{(k+1),L}(p) = V^{-1}(p)\, D_{(\eta)}(p)$$

$$V(p) = D_{(\eta)}(p) \sum_l D_{(\eta)}^{-1}(p+l) \prod_{\mu=1}^{d} \left| \mathscr{C}_\mu(p+l)\, \mathscr{C}_{k,\mu}\left(\frac{p+l}{L^k}\right) \frac{\partial_\mu^L(p+l)}{\partial_\mu^\eta(p+l)} \right|^2$$

$V(p)$ analytic in T_c, $p \in (-\pi/L, \pi/L]^d$, $l = 2\pi m/L$, $m \in \mathbf{Z}^d$, $p + l \in (-L^k\pi, L^k\pi]^d$,

$$|V(p)| < c, \qquad |V^{-1}(p)| < c', \qquad p \in T_c$$

(c) $D_{(\eta)}(p)$ analytic in T_c, $|D_{(\eta)}(p)| < c$; $D_{(\eta)}^{-1}(p+l)$ analytic in T_c, for $l \neq 0$,

$$|D_{(\eta)}^{-1}(p+l)| < c(1 + |p+l|)^{-1}$$

In the proof we use the following lemmas (already established in Section III of ref. 5).

Lemma 4.2. For $p \in T'_c$, $K(p)$, $M(p)$, and $[\sin(p/2)]/[\eta^{-1} \sin(\eta[p+l]/2)]$ are analytic and have the following bounds:

$$\text{(a)} \quad \left| \frac{\sin(p_\mu/2)}{\eta^{-1}\sin[\eta(p_\mu + l_\mu)/2]} \right| \leqslant c(1 + |l_\mu|)^{-1}$$

$$\text{(b)} \quad |K(p+l)| < c(1 + |l|)$$

$$\text{(c)} \quad |M(p+l)| < c(1 + |l|)$$

$$T'_c = \{p\colon \operatorname{dist}(p_1, (-\pi, \pi]) < \pi/2,\ (p_2, \ldots, p_d) \in (-\pi, \pi]^{d-1}\}.$$

Lemma 4.3. For $p \in T_c''$, $(K^2+M^2)^{-1}(p+l)$ is analytic and

$$|(K^2+M^2)(p'+l)| \geqslant c(1+|p'+l|)^2, \qquad l \neq 0, \quad p' \in (-\pi, \pi]^d$$
$$|(K^2+M^2)^{-1}(p+l)| \leqslant c(1+|l|^2)^{-1}$$

where $T_c'' \equiv \{p \mid \operatorname{dist}(p_1, (-\pi, \pi]) \leqslant r, (p_2,..., p_d) \in (-\pi, \pi]^d\}$, r defined below.

Lemma 4.4. (a)

$$\left|\frac{\partial_\mu^1(p)}{\partial_\mu^\eta(p)}\right| \geqslant \frac{2}{\pi}, \qquad \left|\frac{\partial_\mu^1(p+l)}{\partial_\mu^\eta(p+l)}\right| < \frac{\pi}{2}\frac{|p_\mu|}{|p_\mu+l_\mu|}, \qquad l_\mu \neq 0, \quad p \in (-\pi, \pi]^d$$

(b)

$$|K(p)| > c\,|p|; \qquad \eta\,|p| > \pi/2, \qquad p \text{ real}, \quad p \in (-\pi, \pi]^d, \quad c = (2/\pi)^2$$

(c)

$$c'q^2 < K^2(q) + M^2(q) < cq^2; \qquad \eta\,|q| \leqslant \pi, \quad q \text{ real}$$

$$K^2(p+l) + M^2(p+l) \geqslant \frac{1}{d\pi^2}|l|^2$$

$$l = 2\pi m, \qquad m \in \mathbf{Z}^d, \qquad p + l \in (-L^k\pi, L^k\pi]^d$$

The proof of Lemma 4.3 considers the result on the reciprocal of an analytic function: for $f(z)$ analytic in $|z| \leqslant R$ with $\sup_{|z| \leqslant R} |f(z)| < M$ and $f(0) = m$, it follows that $1/f(z)$ is analytic in $|z| \leqslant r \equiv |m| R/4M$ with $|1/f(z)| < 2/|m|$. Hence, the upper bound for $(K^2+M^2)^{-1}$ (details in ref. 5). The proofs of Lemmas 4.2 and 4.4 are elementary (single estimates).

Now we turn to the following proof.

Proof of Theorem 4.1. Assumption (a). For the upper bound on $U(p)$ we use

$$\mathscr{C}_\mu(p+ip') = 1 + \delta\sin(p_\mu + ip'_\mu) = \exp[\delta\sin(p_\mu + ip'_\mu)] + \mathcal{O}(\delta^2)$$

Hence [note that δ and p'_μ are small ($\ll 1$)],

$$\left|\mathscr{C}_{k,\mu}\left(\frac{p+l+ip'}{L^k}\right)\right| \leqslant \left|\exp\left\{\delta\sum_{j=1}^{k}\sin\left(\frac{p_\mu+l_\mu+ip'_\mu}{L^j}\right)\right\} + \mathcal{O}(\delta^2)\right|$$
$$\leqslant \exp\left[2\delta\sum_{j=1}^{k}\sin\left(\frac{p_\mu+l_\mu}{L^j}\right)\right] + \mathcal{O}(\delta^2)$$

for large L and small p' (the sum is bounded by 3). With this bound and Lemmas 4.2 and 4.3, separating the $l=0$ term, we obtain

$$|U(p)| \leqslant c + c \sum_{l \neq 0} \frac{[|K(p)| + |M(p)|][|K(p+l)| + |M(p+l)|]}{|(K^2 + M^2)(p+l)|}$$
$$\times \prod_{\mu=1}^{d} \left| \frac{\sin(p_\mu/2)}{\sin[\eta(p_\mu + l_\mu)/2]} \right|^2 \frac{1}{(\eta/2)^{-2}}$$
$$\leqslant c + \sum_{l \neq 0} \prod_{\mu} \frac{c}{(1 + |l_\mu|)^2} < c$$

To get the lower bound we analyze the $k \to \infty$ limit of the expression for $U(p)$ (similar calculations follow for finite k)

$$U(p) = i\gamma \cdot p \sum_{l \in 2\pi \mathbf{Z}^d} \frac{-i\gamma \cdot (p+l)}{(p+l)^2} \prod_{\mu} \left| \mathscr{C}_{*,\mu}(p+l) \frac{e^{ip_\mu} - 1}{p_\mu + l_\mu} \right|^2$$
$$\equiv i\gamma \cdot p \times -i\gamma \cdot f(p)$$

where $\mathscr{C}_{*,\mu}(p+l) \equiv \lim_{k \to \infty} \mathscr{C}_{k,\mu}((p+l)/L^k)$.

We note that $U(p) \xrightarrow[p \to 0]{} 1$, and so it is necessary to investigate the expression only for p away from zero. We write

$$\mathscr{C}_{*,\mu}(p+l) = \exp\left\{ \delta \sum_{j=1}^{\infty} \sin\left(\frac{p_\mu + l_\mu}{L^j} \right) \right\} + \mathcal{O}(\delta^2)$$
$$\equiv \exp\{\delta h(p_\mu + l_\mu)\} + \mathcal{O}(\delta^2)$$

and hence

$$\gamma \cdot f(p) = \sum_{l} \frac{\gamma \cdot (p+l)}{(p+l)^2} \prod_{\mu=1}^{d} \left| \exp\{\delta h(p_\mu + l_\mu)\} \frac{e^{ip_\mu} - 1}{p_\mu + l_\mu} \right|^2 + \mathcal{O}(\delta^2)$$

For $p_\mu = \pi$ (where U vanishes with the usual average, see Section 3),

$$f_\mu(p) = \sum_{l_\nu \neq l_\mu} \prod_{\nu \neq \mu} \left| \exp\{\delta h(p_\nu + l_\nu)\} \frac{2}{p_\nu + l_\nu} \right|^2 4 \sum_{l_\mu} \frac{\exp\{2\delta h(\pi + l_\mu)\}}{\pi + l_\mu}$$
$$\times \frac{1}{(\pi + l_\mu)^2 + \sum_{\nu \neq \mu} (p_\nu + l_\nu)^2} + \mathcal{O}(\delta^2)$$

The l_μ sum may be written as

$$\sum_{n=1}^{\infty} \frac{1}{n\pi} \frac{2 \sinh[2\delta h(n\pi)]}{(n\pi)^2 + b^2} \geqslant c \sinh \delta, \qquad b^2 = \sum_{\nu \neq \mu} (p_\nu + l_\nu)^2$$

where we used that $h(x) = -h(-x)$. For $0 < p_\mu < \pi$, we also get

$$\sum_{n=0}^{\infty} \left\{ \frac{\exp\{2\delta h(n\pi + p_\mu)\}}{[(n\pi + p_\mu)^2 + b^2](n\pi + p_\mu)} - \frac{\exp\{-2\delta h(n\pi + 2\pi - p_\mu)\}}{[(n\pi + 2\pi - p_\mu)^2 + b^2](n\pi + 2\pi - p_\mu)} \right\} \geqslant c \sinh \delta$$

Similar analysis follows for negative p. For complex values $p + ip'$, with p' small enough ($p' \simeq \delta$), we still have

$$|\mathscr{C}_*(p_\mu + l_\mu + ip'_\mu)| = \exp\{\delta h(p_\mu + l_\mu)\} + \mathscr{O}(\delta)^2$$

which leads to the same bound.

Since $U^{-1}(p) = i\gamma \cdot f(p)/f^2(p) \times -i\gamma \cdot p/p^2$, we obtain $|U^{-1}(p)| < c/\sinh \delta$, proving part (a) of the theorem.

Part (b) is similar, and part (c) comes from Lemmas 4.2 and 4.3. ∎

Finally, the theorem stated in Section 2 follows from the Theorem 4.1 and Lemma 4.1.

5. INTERACTING FERMIONS AND FINAL COMMENTS

Using the RG transformation proposed here, we shall obtain a formula for the generating function of interacting fermions similar to that for the bosonic case (2.4), i.e., written in terms of two propagators P_n and G_n (given by $\sum_j c_j M_j \Gamma_j M_j^\dagger$), an irrelevant potential V_n, and fields related to the minimizers M_n and local effective interactions D_n. In fact, starting with an action such as four fermions plus the Dirac action (properly written) and applying the RG transformation to the generating function, following procedures similar to those considered for the scalar case (see Section 2 and refs. 3 and 4), i.e., separating the quadratic part and using the orthogonal property, after n steps we get

$$Z(\bar{h}, h) = c \exp\{\tfrac{1}{2}(\bar{h}, P_n h)\} \times \int \exp\{-V_n(M_n \bar{\psi} + G_n \bar{h}, M_n \psi + G_n h) - \tfrac{1}{2} b_n(\bar{\psi}, D_n \psi)\}\, d\bar{\psi}\, d\psi$$

with simple expressions for P_n and G_n (such as $\sum_j c_j M_j \Gamma_j M_j^\dagger$), and uniform exponential decay for M_n and D_n, that is, a formalism which shall make easier the analysis of correlation functions.

Another question to be investigated is the connection between wavelets and the structures associated with the scale decomposition as pointed out in Section 2.[3,4]

As a final comment, we emphasize that the RG formalism is considered here only as a useful technical tool, a formalism related to a scale decomposition: we do not assume it as a map "from Hamiltonians to Hamiltonians" (see ref. 6 for problems with such an assumption). Due to properties such as orthogonality between scales, the RG mechanism shall make easier the study and control of the physical correlation functions describing the initial models, although mapping them on "strange" systems (the transformation breaks unitarity). It is also worth remembering that the "exotic" perturbation in the transformation does not change the effective actions at small momenta, i.e., large distances.

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